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The Chord Distribution for a Right Circular Cylinder

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Kellerer's integral expression for the differential chord distribution for a right circular cylinder is evaluated numerically after first reducing to Carlson's symmetric elliptic integrals and standard integrals. This distribution has been useful in microdosimetry and currently is being used in single event upset calculations.

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THE CHORD DISTRIBUTION FOR A RIGHT CIRCULAR CYLINDER

In the evaluation of the effects of an omnidirectional fluence on a sensitive volume the chord distribution is an appropriate tool. It expresses the frequency of occurrence of chords of a given length for a known convex volume (in the three dimensional case). In particular in single event upset calculations one often calculates the upset rate of an omnidirectional fluence passing through the sensitive volume of a memory cell by using the cosmic ray approximation, which makes use of the chord distribution of the sensitive volume.

This report, after reducing the differential chord distribution for a right circular cylinder to Carlson's elliptic integrals and standard integrals, exhibits the evaluation program and numerical and graphical results. Various checks support the validity of the evaluation and precision is discussed. The chord distribution is found to exhibit a cusp and a jump. A plausibility argument is offered for the existence of these features and an analysis of consequences in signal event upset calculations is made.

In Reference 1, Kellerer points out some basic distinctions in probability measure of uniform distributions in two and three dimensions, establishes some background and derives: 1) the path length distribution in a convex body for finite track lengths and 2) the chord distribution in general cylinders given the chord distribution for the cross-section. The present interest is only in the circular case of the latter and further in the density distribution, Kellerer's formula 56:

$$c(s) = \frac{8}{\pi(d+2h)} \left\{ \int_a^1 \left[\left(\frac{h}{\sqrt{1-x^2}} - s \right) x^3 f(sx) + 2x^2 F(sx) \right] dx + \frac{h^2}{s^3} H(s-h) \int_{sa}^{\infty} F(x)dx \right\} \qquad (1)$$

where

$$F(t) = \sqrt{1 - t^2/d^2}, \; f(t) = t/d\sqrt{d^2 - t^2}, \; a = H(s-h)\sqrt{1 - h^2/s^2}$$

and H is the step function ($H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x < 0$). Here d is the diameter of the base, h is the height, and s is the chord length. Also this expression is normalized. Kellerer has given here the expression for a general convex base which has a known chord distribution F (or known f, the latter being the density distribution and $f = -F'$). Note $F, f = 0$ for $t > d$. (In the theory of generalized functions one regularizes these conditions at $t = d$ by writing $F \to HF$ so $f \to Hf + \delta F$, δ being the delta function so one has $H'(x) = \delta(x)$.) Writing one expression and separating into terms indexed by the power of h, one has

$$c(s) = \frac{8}{\pi d(d+2h)} \left[\int_a^1 \left(\frac{hsx^4}{\sqrt{(1-x^2)(d^2-s^2x^2)}} + \frac{2d^2x^2 - 3s^2x^4}{\sqrt{d^2 - s^2x^2}} \right) dx + \frac{h^2}{s^3} H(s-h) \int_{sa}^{d} \sqrt{d^2 - x^2}\, dx \right] \qquad (2)$$

$$= c_1 + c_0 + c_2.$$

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The last two terms are standard integrals but the first should be recognized as an elliptic integral. One notes the maximum chord exceeds d and that at $s = d$ at the upper limit the two square root poles coalesce to form what would appear to be a non-integrable singularity. That this is only appearance may be seen by applying the delta function substitution in the above parenthetical note. Even if one were inclined to try direct numerical evaluation, he would not be surprised when numerical problems arose while trying to evaluate near this pole. The fact that this feature is a standard one for an elliptic integral should be sufficient incentive to pay the penalty of the algebra necessary to reduce this integral to standard elliptic integrals. After this penalty is paid the pole causes no further problem.

Still this algebra is sufficiently onerous that it provides the main incentive for this report. Having once been forced to do this work, one has good reason not to lose the technique. In addition we were forced by the circumstance of having only a single software source [2] to use the modern listing of standard elliptic integrals advanced by Carlson [3] rather than the Legendre standard forms. This was fortunate as Carlson's method takes full advantage of the considerable symmetries in elliptic integrals, resulting in certain evaluation advantages.

In c_1, $s = d$ is a branch point requiring a change in form. This is done by the change of integration variable, $y = sx/d$. The resulting integral has the same form provided one changes the parameter's definition at the branch point: $m = s^2/d^2$ for $s < d$, $m = d^2/s^2$ for $s > d$. Then for $s > d$ one obtains

$$c_1(s) = \frac{8h}{\pi d(d + 2h)} m^2 \int_{sa/d}^{1} \frac{y^4 dy}{\sqrt{(1 - my^2)(1 - y^2)}} . \tag{3}$$

Thus both branches of this elliptic integral can be reduced using the same integral form. To aid the reduction, use of a zero limit may be achieved by $\int_a^1 = \int_0^1 - \int_0^a$. In the following the integration variable is transformed by $y = -x^2$,

$$R_Y(u,m) = \int_0^u \frac{x^4 dx}{\sqrt{(1 - x^2)(1 - mx^2)}} = \frac{1}{2} \int_{-u^2}^{0} \frac{(-y)^{3/2} dy}{\sqrt{(1 + y)(1 + my)}} . \tag{4}$$

Then using SFAM formula 8.1-1 with the substitutions $a = 1$, $a' = 5/2$, $y = 0$, $x = -u^2$, $z_i = 1$, $i = 1,2,3$, $w_1 = 1$, $w_2 = m$, $w_3 = 0$, $b_{1,2} = 1/2$, $b_3 = 5/2$, one has

$$R_Y(u,m) = \frac{1}{2} B\left(1, \frac{5}{2}\right) u^5 R_{-\frac{5}{2}}\left(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, 1 - u^2, 1 - mu^2, 1\right) = \frac{1}{5} u^5 R_{-\frac{5}{2}} \tag{5}$$

since the beta function B, evaluates to 2/5. The R function is related to the hypergeometric series and is used by Carlson to present a unified treatment of many special functions. This function is homogeneous: simultaneous permutation of the first three and last three arguments leaves it unchanged. The local software library [2] includes the following standard elliptic integrals

$$R_F(z) = \frac{1}{2} \int_0^\infty dt\ [(t + z_1)(t + z_2)(t + z_3)]^{-\frac{1}{2}} = R_{-\frac{1}{2}}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, z\right) \tag{6}$$

$$R_D(z) = \frac{3}{2} \int_0^\infty dt [(t + z_1)(t + z_2)(t + z_3)^3]^{-\frac{1}{2}} = R_{-\frac{3}{2}}\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, z\right) \tag{7}$$

where z is understood as a vector, $z = (z_1, z_2, z_3)$ and the integral representations are from SFAM formula 8.3-1. The intention then is to reduce R_Y to R_F, R_D and polynomials following SFAM paragraph 9.3. Applying the formula of SFAM exercise 5.9-8 with the substitutions $t = -3/2$, $c = 5/2$, $w_1 = 3/5$, one has

$$\frac{3}{5}(z_1 - z_2)(z_1 - z_3) R_{-\frac{5}{2}}\left(\frac{5}{2}, \frac{1}{2}, \frac{1}{2}, z\right) = z_2 z_3 R_{-\frac{5}{2}}\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, z\right)$$

$$+ \left(z_1 - \frac{2}{3}(z_2 + z_3)\right) R_{-\frac{3}{2}}\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, z\right) - \frac{2}{3} R_{-\frac{1}{2}}\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, z\right) \tag{8}$$

The second term is an R_D but the other terms need further reduction. For the first term one notes $c = a$ so $a' = 0$ and applies SFAM formula 6.6-5 to obtain

$$R_{-\frac{5}{2}}\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, z\right) = (z_1^3 z_2 z_3)^{-\frac{1}{2}} \tag{9}$$

For the third term apply SFAM formula 5.9-7 with $t = -1/2$, $b_1 = 1/2$, $c = 3/2$ and solve for the $R_{-\frac{1}{2}}$ term to obtain

$$R_{-\frac{1}{2}}\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, z\right) = \frac{3}{2} R_F(z) - \frac{1}{2} z_1 R_D(z) \tag{10}$$

Then substituting these results into (8) with $z_1 = 1$, $z_2 = 1 - u_2$, $z_3 = 1 - mu^2$, and then into (5), one has

$$R_Y(u, m) = \frac{u}{3m}\left[\sqrt{(1 - u^2)(1 - mu^2)} + \frac{2}{3}(m + 1)u^2 R_D(z) - R_F(z)\right] \tag{11}$$

When substituting from (6) and (7) the homogeneity property has been used to permute arguments. In case $u = 1$ here the elliptic integrals are complete. Defining R_X by $\frac{u}{3m} R_X = R_Y$, one has for $s < d$

$$c_1(s) = \frac{8h}{\pi(d + 2h)s} \frac{u}{3} R_X(u, m)\Bigg|_{u=a}^{u=1} \tag{12}$$

and for $s > d$

$$c_1(s) = \frac{8hd}{\pi(d + 2h)s^2} \frac{u}{3} R_X(u, m)\Bigg|_{u=sa/d}^{u=1} \tag{13}$$

One notes that if $s < h$, $a = 0$ and the lower limit term vanishes in both. This condition is a second branch point in the range of s.

The last term is integrable by standard methods

$$c_2(s) = \frac{4h^2 d}{\pi(d + 2h)s^3} H(s - h) \left[x\sqrt{1 - x^2} + \sin^{-1} x \right]_{sa/d}^{1} \tag{14}$$

and the remaining term with a little more work yields

$$c_0(s) = -\frac{d^3}{\pi(d + 2h)s^3} \left[x\sqrt{1 - x^2}\,(6x^2 + 1) - \sin^{-1} x \right]_{sa/d}^{b} \tag{15}$$

where $b = s/d$ for $s < d$ and $b = 1$ for $s > d$, so the upper limit has a branch point at $s = d$ and the lower has one at $s = h$. It is useful to define the auxiliary function

$$R_A(f, x) = x\sqrt{1 - x^2}\, f(x) - \sin^{-1}(x) \tag{16}$$

Then combining (14) and (15), one has

$$c_{0,2}(s) = \frac{d}{\pi(d + 2h)s^3} \left[d^2 R_A(6x^2 + 1, x) \Big|_{sa/d}^{b} - 4h^2 H(s - h) R_A(-1, x) \Big|_{sa/d}^{1} \right]$$

The branch structure is the same as for c_1: the lower limit contributes nothing when $s < h$. Furthermore the $s < h$ term, the upper limit, is part of the total for $s > h$. Both branches, $s < d$ and $s > d$, have this same structure. This structure is utilized in the evaluation, program K3CRDN, listed in the appendix. The programming is straight forward except that this form of the statement function RX assumes SQRT(0.) returns zero. Also use of double precision has been made to avoid round-off error near $s = 0$.

This routine checks normalization and average chord length also. The probability (or integral) distribution is defined

$$C(s) = \int_s^\infty c(x)\, dx \tag{18}$$

and the normalization condition is $C(0) = 1$. Furthermore there is a theorem, sometimes known as the Cauchy theorem [5], that three dimensional chord distributions of convex bodies have an average chord length, $\bar{s} = 4V/S$, where V is the volume and S, the surface area. For a right circular cylinder this is

$$\bar{s} = 2hd/(d + 2h) \tag{19}$$

Five runs of K3CRDN are also exhibited in the appendix for different values of the aspect ratio, h/d, giving the sum over the density to check normalization, the average chord length calculated by

$$\bar{s} = \int_0^\infty s c(s) ds \tag{20}$$

and that obtained from (19) for comparison. These runs were all for 2000 bins (MS) but comparing series of runs for fixed d and h with number of bins increasing exhibited convergence. The accuracy obtained of 3 to 4 figures is acceptable for a 32 bit machine except in the case for $h/d = 10$, where the norm error is 0.2 percent. This is possibly improvable by resort to double precision should the need arise. However neither limit $d \to 0$ or $h \to 0$ exists for the final form so one might expect problems with either very high or very low aspect ratio.

A slightly modified version of K3CRDN has been run to produce the graphs in Fig. 1, showing results for the three most moderate aspect ratios: 0.5, 1.0, and 2.0. Note that the jump, in both cases where it is isolated (the case $h = d = 1$ may be considered degenerate in that the cusp absorbs the jump), is associated with the chord $s = h$, while the cusp is associated with $s = d$. The values of h and d were set at $2/\sqrt{5}$ and $4/\sqrt{5}$ for the high and low aspect ratios and $\sqrt{2}$ in the other case. All three cases have Max(s) $= 2$.

The jump and cusp are interesting features from several standpoints. The differential chord distribution for a rectangular parallelopiped (RPP) also exhibits jumps, three if all three dimensions are different [4]. Having observed these features one can propound reasons for their occurrence, carefully couching them in the language of geometric probability [6] to mask any suggestion of hand waving. The basic uniform measure for chord distributions in three dimensions is four dimensional. These four dimensions correspond to points on any plane plus the element of solid angle at each such point. In this qualitative argument, one need only point out that when a particular chord length has an extra dimension available to it, that length must occur with higher probability in the distribution. In the case of the RPP one sees that the occurrence of opposite parallel faces provides this extra dimension for that length equal to the distance between the two faces. Moreover one sees that this length may be approached continuously from above by variation of solid angle but not from below. That is, a chord produced by displacing a perpendicular chord by a small angle always makes a longer chord. In other words this feature indeed has the characteristic necessary to provide an "up" jump in the chord distribution. In the case of the circular cylinder one sees that "opposing parallel faces" must be expanded to the more inclusive term, "opposing parallel elements," remembering the elements of a cylinder are the straight lines constituting the cylindrical surface. Furthermore one sees that there is a continuous approach from below as well as from above. The length of a diameter is approached from below by angular displacements perpendicular to the elements and from above by displacements parallel to the elements. Hence opposing parallel elements have a cusp signature in the chord distribution.

The cusp and jump are also of interest for, shall we say, their mildly singular character. The cusp is really singular but "mildly" so, as it is integrable. Having seen above that the cause of these features is opposing parallel elements, one also sees that such features are pervasive among typical sensitive volumes in memory cells. While a depletion region (that is, a sensitive volume) is defined in part electrostatically and thus must have rounded corners, such rounding is likely to be mainly local, leaving still significant areas of opposing parallel elements. Indeed while this electrostatic effect tends to ameliorate parallel faces, it may enhance cylindrical parallelism. It seems there is reason to expect physical consequences from these features. In the evaluation of the upset rate in space the coincidence of one of these mildly singular features with a sudden jump in cosmic ray fluence at a particular LET should provide an especially dramatic onset. Of itself this feature is not qualitatively distinguishable because already jumps in the LET spectrum occur. However one prominent evaluation method, Monte Carlo, would miss enhanced onset from this source because it is incapable of evaluating mildly singular features.

Some limit considerations influenced the evaluation while others did not. It seems worthwhile to mention them for their possible affect on modifications. The jump is finite and is no problem. No provision was made for the case that a bin center point might fall on the cusp, which is a singularity. This point is simply not evaluated in the direct coding. However in the evaluation of the two elliptic integral library functions, a stop occurs if both the first two arguments are zero. This could only happen if a bin center fell on a cusp within computer precision. A weak attempt to avoid this was made by dividing bins irrationally with respect to $s = d$ or h by using maximum chord length. Since this

would have imposed unusual input restrictions, cases of rational ratios d or h to Max(s) have been evaluated and no problem was encountered. The case where one formed bins by dividing d or h instead of Max(s) has not been investigated. In fact an investigation of the $s \to d$ limit when $h = d$ was made using known limit properties of the Legendre forms of elliptic integrals. As indicated earlier this contributed little to the results except, perhaps, a familiarity with problems avoided. The distribution end points are easily evaluated. If $s_m^2 = d^2 + h^2$,

$$c(s_m) = 0 \tag{21}$$

$$c(0) = 16/(3\pi(d + 2h)) \tag{22}$$

the latter arising as a limit of the c_0 term. Thus evaluation near $s = 0$ should be avoided without making special provision. In Fig. 1, the three graphs, in order of increasing aspect ratio, begin at 0.474, 0.400, and 0.380, respectively, according to (22).

Perhaps it should be remarked that the rigorously proper mathematical milieu for some of the manipulations above is non-standard analysis since for completeness delta functions are needed. That it was nearly possible to carry through this analysis without reference to delta functions is interesting in itself. The one reference, page 2 and 1, was only to point out that by using one, a difficulty can be seen not to be a problem. Probably the reason this short cut was feasible is that the delta function fortuitously integrates to zero. This is proven by the following exercise.

The same methods as above were applied to evaluate Kellerer's integral chord distribution for a circular cylinder. An ambiguity arose as to the constant of integration applicable for the $s < d$ branch which is related to the choice of the use of arccosine or arcsine in the evaluation of one of the standard integrals. This ambiguity was easily resolved by enforcing the normalization condition, following equation 18. At the same time this results in continuity at $s = d$. This proves the delta function makes no contribution because of contributing one would show as a jump at $s = d$. No ambiguity arises for the $s > d$ branch because it approaches zero at maximum s due to the coalescence of the limits of integration.

The integral distribution, evaluated directly, agreed entirely with the numerical integral of the differential distribution thus providing an additional confirmation of these evaluations. The direct integral is available but is not included here because integrating the differential is just as efficient whenever the complete distribution is being evaluated. Furthermore the integral algorithm seems to be slightly more sensitive to the need for higher precision at small s.

This same evaluation using Legendre elliptic integrals has been published by *U.* Mäder [7] along with a number of related results. For example, the infinite limit of both d and h are obtained and the latter gives our c_1 term. Graphical results appear to be the same after renormalizing to account for the use of different dimensions. The author wishes to thank A. M. Kellerer for this reference.

References

1. A. M. Kellerer, "Considerations on the Random Traversal of Convex Bodies and Solutions for General Cylinders," Radiation Research 47, 359 (1971).

2. Proprietary software marketed by Numerical Algorithms Group, Inc., 1101 31st Street, Suite 100, Downers Grove, IL 60515, VAX version Mark 10. It may be worth remarking that since

this work began two other major software sources have offered the Carlson form rather than the Legendre form. Probably this choice is based on superior numerical methods.

3. B. C. Carlson, "Special Functions of Applied Mathematics," Academic Press (1977), referred to as SFAM.

4. W.L. Bendel, "Length Distributions of Chords Through a Rectangular Volume," NRL Memorandum Report 5369, July 3, 1984.

5. This theorem is proved in both references 1 and 6.

6. M. G. Kendall and P. A. P. Moran, "Geometric Probability," Hafner, 1963.

7. U. Mäder, "Chord Length Distributions for Circular Cylinders," Radiation Research 82, 454 (1980).

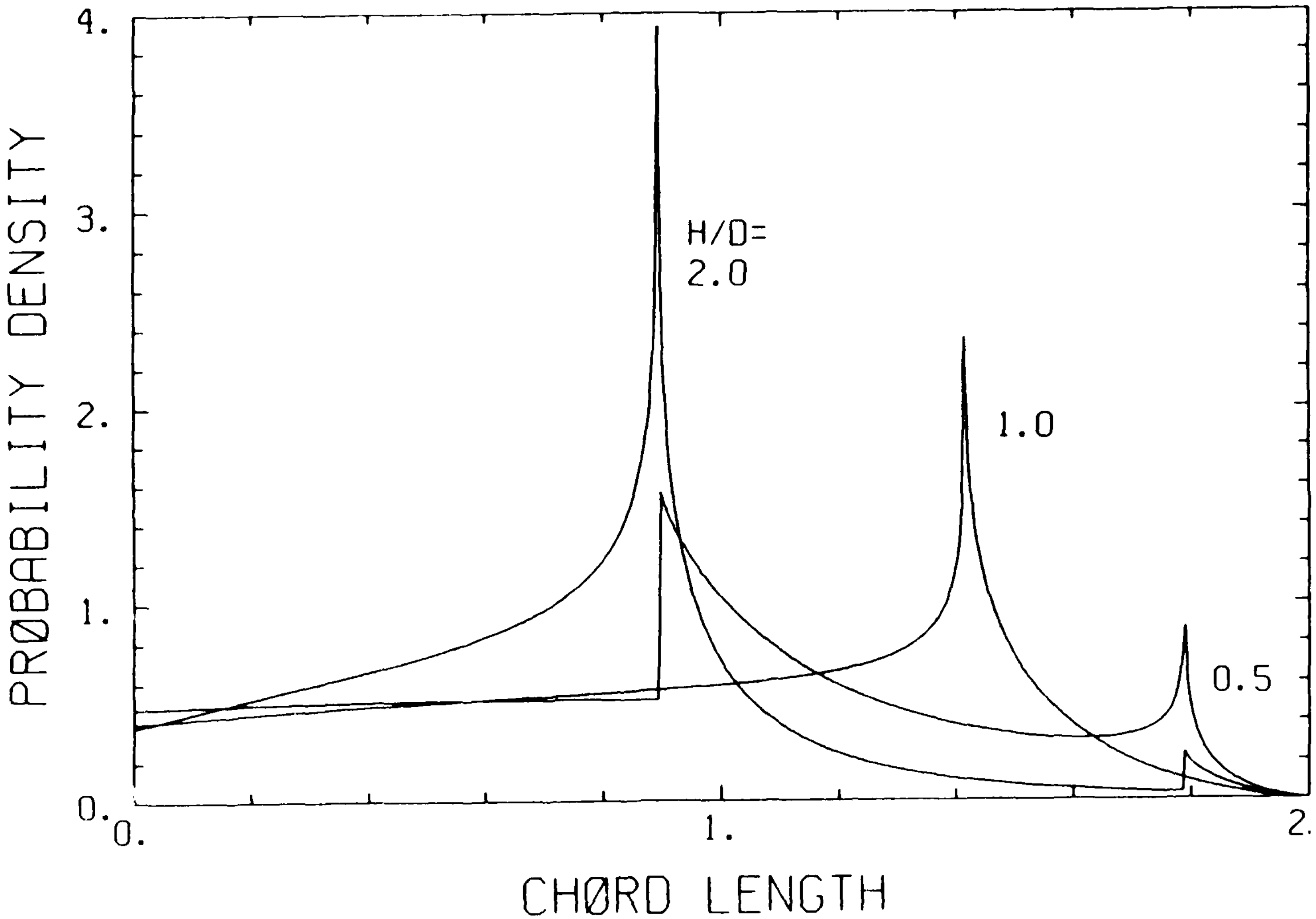

Fig. 1 — The chord length density distributions for 3 cylinders of varying aspect ratio, height/diameter

```
      PROGRAM K3CRDN
C() IS THE DIFFERENTIAL CHORD DISTRIBUTION OF A CYLINDER OF DIAMETER D &
C HEIGHTH H BY EVALUATING KELIERER'S FORMULA, RAD RESEARCH 47, 373 (1971). RF
C & RD ARE CARLSON'S SYMMETRIC ELLIPTIC INTEGRALS OF THE FIRST & SECOND KINDS
C THE ABSCISSA (NOT OUTPUT) IS THE MAX CHORD DIVIDED INTO MS BINS  P2 IS PI/2
      DIMENSION C(2000)
      REAL*8 EM
      DATA IFAIL/0/
      RD(U,V)=S21BCE(U,V,1.,IFAIL)
      RF(U,V)=S21BBE(U,V,1.,IFAIL)
      RA(F,U)=U*SQRT(1.-U*U)*F-ASIN(U)
      RX(U,V,W)=SQRT(U*V)+W*RD(U,V)-RF(U,V)
      P2=X01AAE(1.)*0.5
      TYPE 1
1     FORMAT(' ENTER D,H,MS,KS,FO:'$)
      ACCEPT *,D,H,MS,KS,FO
      IF(MS.GT.2000) STOP 101
      D2=D*D
      H2=H*H
      CM=SQRT(D2+H2)
      DS=CM/MS
      S=-0.5*DS
      FO=D+H+H
      F1=0.5/(P2*FO)
      F5=2./3
      F2=4.*H*F5
      Z0=1.+H2/D2
      H3=4.*H2
      Z1=0
      A=0.
      F=0.
        DO 4 M=1,MS
      S=S+DS
      S2=S*S
      SD=S/D
      IF(S.GT.D) GOTO 3
      EM=S2/D2
      ES=EM
      ET=(ES+1.)*F5
      F3=F2/S
      F4=1./(ES*SD)
      Z2=1D0-EM
      CD=F3*RX(Z1,Z2,ET)+F4*RA(6.*ES+1.,SD)
      IF(S.LE.H) GOTO 2
      Z3=H2/S2
      Z4=Z0-ES
      A2=1.-Z3
      ZA=SQRT(A2)
      ZU=ZA*SD
      U2=ZU*ZU
      CD=CD-F3*ZA*RX(Z3,Z4,ET*A2)-F4*RA(6.*U2+1.,ZU)
     1 +H3/(SD*S2)*(P2+RA(-1.,ZU))
      GOTO 2
3       EM=D2/S2
      ES=EM
      ET=(ES+1.)*F5
      F3=F2/(SD*S)
      F4=ES/SD
      Z2=1D0-EM
      CD=F3*RX(Z1,Z2,ET)-F4*P2
      IF(S.LE.H) GOTO 2
      Z3=H2/S2
      Z4=Z0-1./ES
      ZA=SQRT(1.-Z3)
      ZU=ZA*SD
      U2=ZU*ZU
      CD=CD-F3*ZU*RX(Z3,Z4,ET*U2)-F4*RA(6.*U2+1.,ZU)
     1 +H3/(SD*S2)*(P2+RA(-1.,ZU))
2     CD=CD*F1
      C(M)=CD
      A=A+CD
4     F=F+CD*S
        F=F/A
      A=A*DS
      SB=2.*D*H/FO
      IF(KS.EQ.0) GOTO 8
      K1=(KS+1)/2
      TYPE 9,(C(M),M=K1,MS,KS)
9     FORMAT(' PROBABILITY DENSITY:'/(X,10F8.3))
8     TYPE 10,A,F,SB
10    FORMAT(' NORM:'F10.7' AVE:'F11.8' COMPARE:'F11.8)
      IF(FO.EQ.1) WRITE(7,7) (C(M),M=K1,MS,KS)
7     FORMAT(1P5E14.6)
      END
```

```
$ R K3CRDN
ENTER D,H,MS,KS,FO:
2.,2.,2000,0,0
NORM: 0.9999697 AVE:   1.333315 COMPARE:   1.333333
$ R K3CRDN
ENTER D,H,MS,KS,FO:
2.,1.,2000,0,0
NORM: 1.0005059 AVE:   0.999779 COMPARE:   1.000000
$ R K3CRDN
ENTER D,H,MS,KS,FO:
1.,2.,2000,0,0
NORM: 1.0005429 AVE:   0.800043 COMPARE:   0.800000
$ R K3CRDN
ENTER D,H,MS,KS,FO:
2.,.2,2000,0,0
NORM: 1.0000465 AVE:   0.333263 COMPARE:   0.333333
$ R K3CRDN
ENTER D,H,MS,KS,FO:
.2,2.,2000,0,0
NORM: 0.9978931 AVE:   0.190463 COMPARE:   0.190476
```